Proof reading of an integration problem

Check the following integration steps. Are there any mistakes that lead to the different answers in Method 1 and Method 2?

Method 1

Evaluate $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

We have the standard integral: $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + c$.

$I = \int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4-(4-x^2)}{\sqrt{4-x^2}}\,dx = 4\int \frac{1}{\sqrt{4-x^2}}\,dx - \int \sqrt{4-x^2}\,dx$

$= 4\sin^{-1}\left(\frac{x}{2}\right) - \left[x\sqrt{4-x^2} - \int x\,d\left(\sqrt{4-x^2}\right)\right]$, integration by parts.

$= 4\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{4-x^2} + \int x\frac{-2x}{2\sqrt{4-x^2}}\,dx = 4\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{4-x^2} - I$

$\therefore 2I = 4\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{4-x^2}$

$\therefore I = 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C \quad \ldots (1)$

Method 2

$$I = \int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

Let $u^2 = 4 - x^2$, $2u\,du = -2x\,dx$, $u\,du = -x\,dx$

$I = -\int \frac{\sqrt{4-u^2}}{u}\,u\,du = -\int \sqrt{4-u^2}\,du$

Let $u = 2\sin t, du = 2\cos t\,dt$

$I = -\int 2\cos t\,(2\cos t\,dt) = -4\int \cos^2 t\,dt = -4\int \left(\frac{1+\cos 2t}{2}\right)dt = -2[t + \sin 2t] + c$

$= -2t - 2\sin t\cos t + c = -2\sin^{-1}\left(\frac{u}{2}\right) - u\sqrt{1-\left(\frac{u}{2}\right)^2} + c = -2\sin^{-1}\left(\frac{u}{2}\right) - \frac{1}{2}u\sqrt{u^2-1} + c$

$= -2\sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + c \quad \ldots (2)$

It seems that the answer in (1) is different from (2), but in fact they differ by a constant.

We like to show : $\boldsymbol{\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right) = \sin^{-1}\left(\frac{x}{2}\right)}$

Consider $\cos\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] = \sin\left[\sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] = \frac{\sqrt{4-x^2}}{2}$

$$\cos^2\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] = \frac{4-x^2}{4}$$

$$\sin^2\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] = 1 - \frac{4-x^2}{4} = \frac{x^2}{4}$$

$$\sin\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] = \left(\frac{x}{2}\right)$$

$$\frac{\pi}{2} - \sin^{-1}\left(\frac{\sqrt{4-x^2}}{2}\right) = \sin^{-1}\left(\frac{x}{2}\right)$$

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